1.3: Nonlinear Waves

In general, as a linear dispersive wave system evolves, each Fourier component with wavenumber k propagates with its own group velocity, and so the system disperses. Then nonlinearity, that is the necessity to take account that the amplitude is finite and not infinitesimally small, typically arises in three scenarios.

(1) **Long waves**: Here $k \to 0$. Because the dispersion relation can be made to satisfy the antisymmetry condition $\omega(k) = -\omega(-k)$ (ensuring real-valued solutions), it follows that when also $\omega(0) = 0$, we have that $\omega = c_0 k + O(k^3)$, and so $c_g = c_0 + O(k^2)$, with **weak dispersion**.
(2) **Wave packets**: Here it is assumed that the wave energy is concentrated around a finite wavenumber k_0 say. Consequently, there is again only **weak dispersion**, and approximately the wave group propagates with a constant group velocity $c_{g0} = c_g(k = k_0)$.
(3) **Resonant wave interactions**: Due to nonlinearity, two linear waves with wavenumbers $k_{1,2}$ say, will interact to form another wave with wavenumber $k_0 = k_1 + k_2$. If the corresponding frequencies are resonant, that is $\omega_0 \approx \omega_1 + \omega_2$ ($\omega_i = \omega(k = k_i)$), then there can be a strong effect.